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Numerical simulations of burst processes in fibre bundles

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Abstract. We discuss a very effective numerical method for simulating fibre-bundle models with equal load-sharing and with local load-sharing. Particular attention is paid to the case of the local load-sharing model, in which the critical load x_c is defined as the average load per fibre that causes the final complete failure. It is shown that $x_c \to 0$ when the size of the system $N \to \infty$. We also show analytically that the power law of the burst size distribution, $D(\Delta) \propto \Delta^{-\xi}$, is approximately correct. The exponent ξ in the local load-sharing case is not universal, since it depends on the strength distribution as well on as the size of the system.

1. Introduction

The statistical properties of the strength of materials with stochastically distributed elements are important in practical applications. A simple classical model in this field is the fibre-bundle model, which has been studied by many authors [1–10]. There are two main types of fibre-bundle models according to their rule of load distribution. One type is the equal load-sharing, or global load-sharing [1], model, and the other one is the local load-sharing model. Consider a bundle of N fibres, stretched at both ends. The strength x of the fibres, i.e. the maximum load the fibres are able to carry, varies from fibre to fibre according to some probability density distribution $p(x)$. The fibres are stretched until they fail. In the global load-sharing fibre-bundle model, as each individual fibre breaks, the load distributes itself equally among the surviving fibres, while the failed fibres carry no load. Hemmer and Hansen [1,5] proved that in this case, the expected number $D(\Delta)$ of the burst of size Δ, in which a total number Δ of fibres break simultaneously, follows asymptotically a power law:

$$D(\Delta) \propto \Delta^{-\xi} \tag{1}$$

with the universal exponent $\xi = 2.5$, for an arbitrary distribution of individual fibre strength. This power law is proved to be exact when $N \to \infty$. It has also been shown [1] that the breakdown process approaches a critical point. This lends strong support to the conjecture that fracture in brittle materials under load is a critical process, i.e. a dynamical process that evolves toward an attractive fixed point, as is typical of self-organized criticality [11]. Another type of fibre bundle is associated with local load-sharing. This type of model is motivated by the fact that the assumption of global load-sharing is often unrealistic. The extreme form for local load-sharing is that all the extra stress caused by a fibre failure is taken up by the nearest neighbours of the failed fibres. So spatial structure is introduced into the fibre-bundle model, while the global load-sharing model has no spatial structure. Hansen and Hemmer [12] studied a one-dimensional case for extreme local load-sharing, and gave some results different from that of the global load-sharing model. In particular, they found that the exponent ξ in the burst size distribution $D(\Delta) \propto \Delta^{-\xi}$ is different from 2.5, the result for the global load-sharing model. By numerical simulation, they found in the local load-sharing model exponents ξ larger than 4, in particular $\xi \simeq 4.5$ for a uniform threshold distribution, when the size of the system N is of the order of 10^3. They left the question 'Is the result $\xi \simeq 4.5$ universal, i.e. under mild restrictions independent of the strength distribution?' unanswered. In order to answer such a question, simulations on a large system with different types of strength distribution should be carried out.

For fibre-bundle models, the usual strategy of computer simulation is to set aside an array of N elements, to record the strength of each fibre. Random numbers x with specified distribution $p(x)$ are preassigned to the fibres as their strength. Then the system evolves according to its dynamical rules. The simulations are restricted by limited computer memory and CPU time, so the system size N cannot be very large. However, it is important to simulate systems of large size in order to exploit the asymptotic behaviour of the model. In order to simulate a system of large size, more effective simulation methods should be used. In this paper, we will discuss in detail a new simulation method, which allows simulations on systems of large size, and which also reduces CPU time expenditure. In section 2, we introduce a kind of random number generator, *ordered random number generator*, the essential ingredient of the new method. Section 3 is devoted to discussing the application of the new simulation method to the global load-sharing fibre-bundle model. In section 4, we discuss in detail the burst process and the simulation of the local load-sharing model. We also try to give some analytical results for the burst size distribution. Some results have been reported previously [7]. Conclusions are given in section 5.

2. Ordered random number generator

The most commonly used random number generator is the uniform one, which generates uniformly distributed random numbers in [0, 1]. Generators of other distribution forms can be constructed by using the uniform one. Here, we present an ordered random number generator, which generates random numbers in an ordered sequence, with a specified distribution. Suppose we need total number N of random numbers x with distribution $p(x)$, in the interval [0, 1] (Any other region can be mapped to [0, 1]). Let us number them by $0 < x_1 < x_2 < x_3 \cdots < x_N < 1$. By using the usual random number generator, we can certainly make N random x's. But the problem is that generally the smallest number is not generated first. The ordered random number generator we proposed is able to give out random numbers in an ordered sequence, from small to large, without loss of randomness.

When x_i is given, we know that there are $N - i$ random numbers in the interval $[x_i, 1]$ and that among these x_{i+1} is the smallest. The probability density distribution of x_{i+1} is

$$q(x_{i+1}) \propto p(x_{i+1}) \left[1 - \frac{\int_{x_i}^{x_{i+1}} p(x)\, \mathrm{d}x}{\int_{x_i}^{1} p(x)\, \mathrm{d}x} \right]^{N-i-1} . \tag{2}$$

In the case of $p(x) = 1$, equation (2) reduces to

$$q(x_{i+1}) \propto \left[1 - \frac{x_{i+1} - x_i}{1 - x_i} \right]^{N-i-1} . \tag{3}$$

When $N - i \gg 1$, noticing that $1 - x_i \approx (N - i - 1)/N$, we have

$$q(x_{i+1}) \propto e^{-N(x_{i+1}-x_i)}. \tag{4}$$

By normalizing the above equation, we get

$$q(x_{i+1}) = B(x_i)e^{-N(x_{i+1}-x_i)} \tag{5}$$

where

$$B(x_i) = \left[\int_{x_i}^{1} e^{-N(x_{i+1}-x_i)}\, dx_{i+1}\right]^{-1}.$$

So the probability that x_{i+1} is no larger than x is

$$y = \int_{x_i}^{x} B(x_i)e^{-N(x_{i+1}-x_i)}\, dx_{i+1}$$

i.e.

$$y = \frac{1 - e^{-N(x-x_i)}}{1 - e^{-N(1-x_i)}}. \tag{6}$$

So we have

$$x = x_i - \frac{1}{N}\ln[1 - y(1 - e^{-N(1-x_i)})]. \tag{7}$$

By now we have constructed the ordered random number generator. We can generate the ordered numbers in the following way: First, set $x_0 = 0$. Next, take a random number y_1 from the usual uniform generator, then use (7) to obtain x_1

$$x_1 = -\frac{1}{N}\ln[1 - y_1(1 - e^{-N})].$$

Then $x_2, x_3, \ldots, x_i$ are obtained in the same way.

$$x_i = x_{i-1} - \frac{1}{N}\ln[1 - y_i(1 - e^{-N(1-x_{i-1})})].$$

Here $\{y_i\}$ are random numbers uniformly distributed in [0, 1].

We should note here that in obtaining (3), we have assumed $N - i \gg 1$. For this reason, the above procedure is reliable only when $N - i \gg 1$. For $N - i \sim 1$ the ordered random number generator might not work properly. But this situation has little effect on our simulations, because we only need the reliable part of the N ordered random numbers. We will explain this point in the following sections.

We have succeeded in obtaining random numbers in ordered sequence with distribution $p(x) = 1$. Suppose $0 < z_1 < z_2 < z_3 < \cdots < z_N < 1$ are N random numbers with uniform distribution in [0, 1]. When N is large

$$i/N \approx z_i.$$

Assume that $0 < x_1 < x_2 < x_3 < \cdots < x_N < 1$ are N random numbers with distribution $p(x)$ in [0, 1]. When N is large

$$i/N \approx \int_0^{x_i} p(x)\, dx.$$

Combining the above two equations, we have

$$z_i = \int_0^{x_i} p(x)\, dx. \tag{8}$$

Using equation (8), we can generate x_i, the ith ordered random number with distribution $p(x)$, through the uniformly distributed random number z_i. In this paper, we mainly consider the distribution form $p(x) = \nu x^{\nu-1}$. For this form of $p(x)$, we simply have $x_i = z_i^{1/\nu}$.

3. Simulation on global load-sharing models

The global load-sharing model has been investigated thoroughly by Hemmer and Hansen [5]. If fibre i is stretched to a length $l = l_0 + x$, its force response is

$$f_i = \begin{cases} \kappa x & \text{if } x < t_i \\ 0 & \text{if } x \geqslant t_i. \end{cases} \tag{9}$$

Here t_i is the failure threshold of this fibre, i.e. the strength of this fibre, and l_0 is the natural length of the fibre. For simplicity, the modulus of the fibre, κ, is assumed to be unity. Let x_k be the ordered sequence of failure threshold t_i: $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_N$. The external load F on the bundle at the moment when the kth fibre is about to break can be expressed as

$$F_k = (N + 1 - k)x_k. \tag{10}$$

Note that the external loads $\{F_k\}$ at which the fibre would fail do not form a monotonically increasing sequence. Thus all values F_k will not be realized in the rupture process. Assume that the external load F has reached a value F_k, and that the subsequent values satisfy $F_{k+j} \leqslant F_k$ for $j = 1, 2, \ldots, \Delta - 1$ while $F_{k+\Delta} > F_k$. In this case, all fibres corresponding to $x_k, x_{k+1}, \ldots, x_{k+\Delta+1}$ fail, without changing the external load, before the bundle again reaches equilibrium. The simultaneous failure of the Δ fibres is called a burst of size Δ. The purpose of the simulation on this model is to find the relation between the expected number of burst, $D(\Delta)$, and the size Δ. Using the ordered random number generator, the simulation can be carried out very easily. First, x_1, the strength of the weakest fibre in the bundle, is obtained from the generator. Then calculate F_1 according to (10) and record F_1 as the current largest F_k. Then x_2 is generated by the generator, and F_2 is calculated. If $F_2 \geqslant F_1$, the current largest F_k is replaced by F_2, then we know that a burst of size 1 occurred. If $F_2 < F_1$, relinquish F_2 and calculate F_3, and so on. In the process, we only need to reserve space in the computer memory for the current largest F_k and for the strength of the current fibre that is about to fail. It is not necessary to consider all the fibres in the system at the same time. So the computer memory required is greatly reduced, and CPU-time expenditure is reduced to only about 2 percent of the usual method. We made some calculations on a system of $N = 500\,000$ fibres using a 'Sun 4 SPARC standard 1+' workstation. The CPU time expenditure per sample is 25.4 seconds. The exponent for the burst size power-law distribution is in agreement with the analytical result in [5].

We also tried the usual method which considers all fibres at the same time. In this case, the system of the largest size, to which the usual method can be used, consists of 50 000 fibres. The CPU time expenditure for such a system by the usual method is 127.8 seconds per sample. Compared with the new method of simulation, this is quite large.

4. Fibres bundle with local load-sharing

4.1. Simulation method

For the fibre bundle model with local load-sharing, the simulation is much more complex than that of global load-sharing because spatial structure is introduced. In the extreme version for the local load-sharing model all the extra stress caused by a fibre failure is taken up by the nearest neighbours of the failed fibres. In the one-dimensional case, the total number N of parallel fibres are mounted equidistantly on a circle, so each fibre has two nearest neighbours. If one fibre fails, its two neighbours will take up its load. If one or both of its nearest neighbours had failed, further neighbours effectively become its nearest neighbours, and so on. For example, if fibre 1 fails, fibre N and fibre 2 will share equally the load of fibre 1 (Note that fibre N is one of the nearest neighbours of fibre 1). If fibre 2 also fails, fibre 3 and fibre N take up the loads of fibres 1 and 2. This model has been discussed previously for a different purpose [10]. Hansen and Hemmer reported recently [12] that this model is *not* in the universality class of the fibre bundles with global load-sharing. It will be shown by our simulations on larger bundles that the exponent ξ for the local load-sharing case is not universal, in the sense that its value depends both on the strength distribution $p(x)$ and on the number N of fibres. At the beginning of the burst process, the weakest fibre breaks first, then if its neighbours cannot stand their load, they will break; if they are strong enough to support the load, the next fibre to break will be the second weakest one, and so on.

Using the above approach, it is sufficient in the simulation to record only the neighbours of failed fibres and the weakest surviving fibre, because the next fibre to break is always among them. Information about these fibres are recorded in an array. Once a fibre fails, information about this fibre will be removed and information about some other fibres should be recorded. Assume that the current weakest fibre has strength x_i. If the newly broken fibre is not the weakest one, we need only to record the strengths of its neighbours. Their strengths must be larger than x_i. However, if the newly broken fibre is the weakest one, we then need two independent random numbers, one which is used to determine the location of the new weakest surviving fibre, and the other one to determine the strength of the new weakest fibre, which should be obtained through the ordered random number generator. Whenever the load on one fibre of the system exceeds 1, which is the maximum strength of the fibres, we know that complete failure of the system occurs. It is in this sense that we claimed in section 2 that we only need the reliable part of the N ordered random numbers (i.e. x_1 through x_i, with $N - i \gg 1$).

With this new method of simulation, CPU time and computer memory expenditure are greatly reduced. For example, using the usual method we can only simulate bundles which consist of at most 50 000 fibres, and the CPU time expenditure for a sample of 50 000 fibres is 1, 406 seconds. Using the new method we can simulate fibre bundles of at least 400 000 fibres. CPU time expenditure is reduced to less than ten percent of the usual method. It should be noted that the two methods give the same results when the system size is not too large, so that both methods are applicable.

In figure 1, we illustrate a typical bursting process of the system. We can see that at the beginning, fibres fail in random places, and leave '*holes*' in the system. Here we call the successively failed fibres a '*hole*'. As the burst events continue to occur, some holes will probably connect to each other and become one large hole. We define Θ as the ratio between the number of 'holes' in the system and the number of burst events that have occurred *just before complete failure of the system*. We also have some statistical data on Θ. The results are shown in figure 2, from which we may expect that $\Theta \to 1$ when $N \to \infty$. Based on this numerical evidence, we assume that before complete failure of the system, almost every hole in the system is caused by a single burst event as the system goes to infinity. Hence, we may mainly consider the burst events which create a new hole without connecting to any other holes.

4.2. *Critical load x_c and its asymptotic behaviour*

Let us denote the total load on the whole system by Nx, where x is the average load per fibre in the bundle. As the load on the system increases, a series of bursts of all sizes

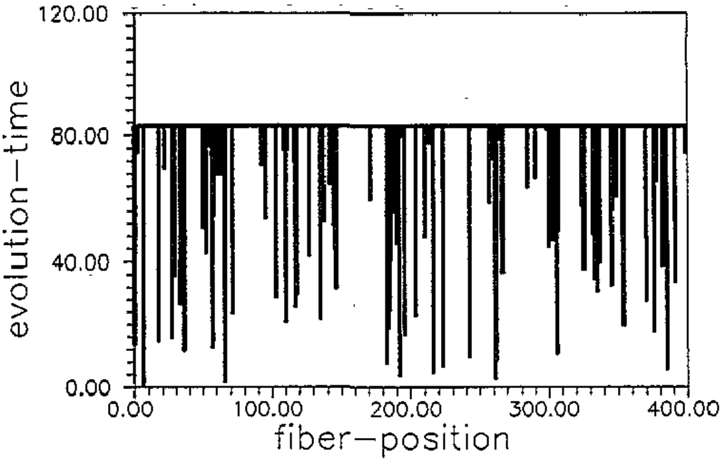

Figure 1. A typical breakdown process of a fibre bundle. White indicates an intact fibre, black a broken one. Evolution 'time' is defined as the number of burst events which occurred. At the end of the process, all fibre positions are marked by black, meaning that all fibres have failed.

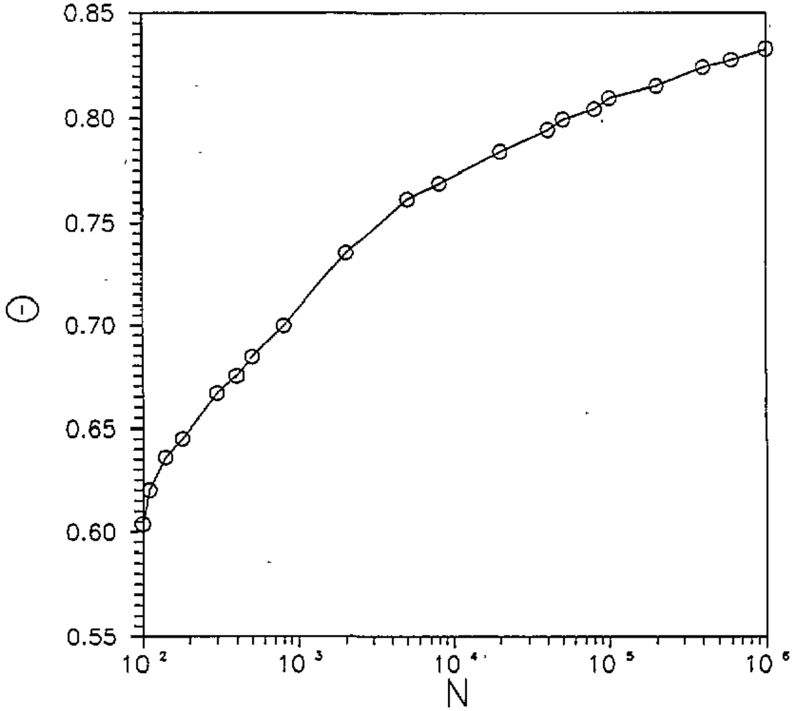

Figure 2. $\Theta \to 1$ when system size N goes to infinity This indicates that almost every '*hole*' in the fibre bundle is caused by a single burst event.

occur. As the average load x approaches a critical value, x_c, a complete failure of the system will result. In the global load-sharing fibre-bundle model, the counterpart of x_c is just x_0 (see [5]), which is also the average load that causes the complete failure. In that case, x_0 satisfies $x_0 p(x_0) = 1 - P(x_0)$ with $P(x) = \int_0^x p(y)\, dy$. So x_0 is dependent on the strength distribution, $p(x)$, but independent of the size N of the system. However, in the present local load-sharing model, x_c is no longer independent of the system size. We found in numerical simulations that the expression

$$\frac{1}{x_c} = a \ln N + b \tag{11}$$

holds for many different strength distributions, where a and b are constants depending on $p(x)$. In figure 3, we show the results for three types of strength distributions: $p(x) = 1$, $p(x) = 2(1-x)$ and $p(x) = 2x$. From the numerical results we can see that (11) is valid for a quite large region of N. When N becomes larger and larger and as $N \to \infty$, we expect equation (11) still to be valid.

Here we argue qualitatively for the validity of equation (11). We restrict ourselves to the case where the strength distribution $p(x)$ is confined in a finite interval. Assume that the strength distribution is uniform, i.e. $p(x) = 1$ for $x \in [0, 1]$. A fibre, fibre 1 say, with strength x_1, is assumed to cause the final complete failure. We may consider only the case in which the failure of fibre 1 creates a new hole. To cause the final failure, at least one nearest neighbour of fibre 1, fibre 2 say, must have strength $x_2 \in [x_1, \frac{3}{2}x_1]$, and one nearest neighbour of fibre 1 or fibre 2, fibre 3 say, must have strength $x_3 \in [x_1, 2x_1]$, and so on. Let r be the smallest integer satisfying the following inequality:

$$\frac{r+2}{2} x_1 \geqslant 1 \tag{12}$$

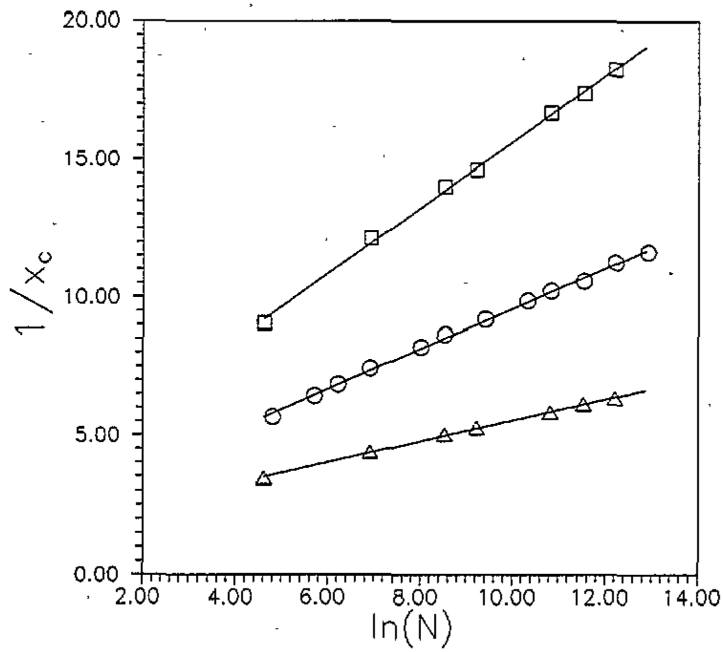

Figure 3. The critical load x_c is a decreasing quantity with increasing fibre number N. $1/x_c$ versus $\ln N$ is shown for three types of strength distribution: ○, $p(x) = 1$; □, $p(x) = 2(1-x)$; △, $p(x) = 2x$.

which means that the load on fibre r is larger than 1, the largest threshold. From (12) we have

$$|\mathrm{d}x_1| = \frac{2}{(r+2)^2}|\mathrm{d}r|.$$

For $|\mathrm{d}r| = 1$, $|\mathrm{d}x_1| = 2/(r+2)^2$. If the r fibres do not connect any hole created before the final failure, the probability that the fibre with strength x_1 causes the final failure is about

$$2^{r-1}\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2 \int_{x_1}^{2x_1} \mathrm{d}x_3 \cdots \int_{x_1}^{\frac{r+1}{2}x_1} \mathrm{d}x_r = (r-1)!\,x_1^{r-1}. \tag{13}$$

The number of fibres with strength in $[x_1 - \mathrm{d}x_1, x_1]$ is $N\,|\mathrm{d}x_1| \simeq 2N/(r+2)^2$. So if x_1 is really the critical load x_c, we should have roughly $(r-1)!\,x_c^{r-1} \cdot N\,|\mathrm{d}x_c| \sim \mathrm{O}(1)$, or

$$\frac{2AN}{(r+2)^2}(r-1)!\,x_c^{r-1} = 1 \tag{14}$$

where A is a constant of order 1. When N is very large and hence x_c is small, r is quite large. Using Stirling's formula and equation (12) we get

$$\frac{1}{x_c} = \frac{\ln N}{2(1-\ln(2-3x_c))} + \left[\frac{\ln A - \ln 2 + 2\ln x_c}{2(1-\ln(2-3x_c))} + \frac{3}{2}\right]. \tag{15}$$

Since $\ln x_c$ changes much more slowly than $1/x_c$, the terms in the bracket in (15) can be considered as a constant, and the validity of equation (11) is qualitatively demonstrated. If the first r broken fibres in the last failure connect to some holes created previously (this case has a very small probability of occurring when $N \to \infty$), equations (12)–(15) should be modified, but (11) is still correct. It is clear that for non-uniform strength distribution $p(x)$, the conclusion is also valid.

A word is in order here about the total strength F_c of the local load-sharing fibre bundle. We notice from (11) that $F_c = Nx_c \sim N/a\ln N \to \infty$ as $N \to \infty$, hence the divergence of F_c is much slower than that of the global load-sharing model.

4.3. Burst size distribution

Equation (11) indicates that $x_c \to 0$ when $N \to \infty$. The fact that $x_c \to 0$ is consistent with the assumption that almost every hole in the system is caused by a single burst. We may hereafter only consider this kind of burst. In addition, as we mentioned before, we mainly consider a strength distribution of the form

$$p(x) = \nu x^{\nu-1} \qquad \text{for } x \in [0, 1]. \tag{16}$$

We first note that the burst event of $\Delta = 1$ is simple and occurs in this way: when a fibre of the strength x_1 cannot stand a load $x \geqslant x_1$, the fibre breaks; then the load is transferred to its two neighbours according to the load-sharing rules; both the two neighbours are strong enough to stand the redistributed loads and they do not break; and the burst stops here. Since in the above process only one fibre breaks, it is thus a $\Delta = 1$ burst event.

For $\Delta > 1$, a burst event can occur in different ways. For example, when $\Delta = 2$, the burst event can occurs in two ways, which could be formally denoted by (1 2) and (2 1) respectively. We call (1 2) and (2 1) the two *configurations* of the bursts $\Delta = 2$. Here the numbers 1 and 2 in the brackets stand for the two failed fibres and also indicate the sequence in which the fibres break. The burst event (1 2) occurs in this way: fibre 1 breaks first and then after load redistribution fibre 2, the right neighbour of fibre 1, breaks too, and then the burst stops. Hereafter we use this convention to write the configurations of the burst events: fibres that are involved in a burst event are numbered according to the sequence in which they break; if in a configuration two fibres are numbered identically this means these two fibres break at the same moment. An example is (2 1 2), which is a burst configuration for $\Delta = 3$. In this burst, fibre 1 breaks first, then after load redistribution, neither of its two fibres is strong enough to stand the load, the two neighbours then break at the same moment. Since in this burst the two neighbours of fibre 1 break at the same moment, they are numbered with the same number 2. Although the two fibres are numbered identically we can still distinguish them through their positions.

It can be calculated that for the the burst events of size $\Delta > 2$ there are

$$\sum_{i=0}^{n} 2^{\Delta-1-2i} \frac{(\Delta - i - 1)!}{i!(\Delta - 2i - 1)!} \tag{17}$$

possible configurations, where $n = [\frac{\Delta-1}{2}]$. As an example we list the total five configurations for $\Delta = 3$ burst. They are: (2 1 2), (1 2 3), (3 2 1), (2 1 3), and (3 2 1).

Suppose we have calculated the probability of each burst configuration. Then by adding up the probabilities of all configurations for the burst of Δ, we can get the expected number $D(\Delta)$ of burst events of size Δ

$$\frac{D(\Delta)}{N} = \sum \text{Prob(configuration)}. \tag{18}$$

Now we explain how to calculate the probability for each burst configuration. We write out the probability of a configuration according to the strength evaluations of the fibres involved in the burst. For example, the probability of the configuration (2 1 2) is

$$P(212) = \int_0^{x_c} \mathrm{d}x_1\ p(x_1) \left[\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2'\ p(x_2') \right]. \tag{19}$$

In the square bracket in the above expression, there are two integrations, each of which corresponds to the strength evaluation of a fibre in the burst. It is obvious that the strength x_1 of fibre 1 must be in the interval $[0, x_c]$, so we have in the above expression the integration $\int_0^{x_c} \mathrm{d}x_1\ \cdots$. The other two fibres numbered with 2 break later than fibre 1, so their strengths must be larger than x_1; after fibre 1 breaks, the load on each of the two neighbours is $3x_1/2$, then the two neighbours break at the same moment, so their strength must be smaller than $3x_1/2$, i.e. $x_1 < x_2 < 3x_1/2$ and $x_1 < x_2' < 3x_1/2$.

In a similar way we write out the probability for the configuration (1 2 3) as

$$P(123) = \int_0^{x_c} \mathrm{d}x_1\ p(x_1) \left[\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \int_{x_1}^{2x_1} \mathrm{d}x_3\ p(x_3) \right]. \tag{20}$$

Although in principle we can calculate the expected number of bursts $D(\Delta)$ by (18), the calculation can only be carried out for small Δ. For large Δ, the total number of configurations is large and makes this kind of calculation very expensive in terms of CPU time. Fortunately we have found a way to solve this problem. The key point of the method *is to classify these configurations properly, then we get a 'hierarchy relation' for calculating* the probabilities of these configurations.

For a given Δ we classify the configurations into three groups: group α, group β and group γ. Group α is the set of configurations that have the form $(i\ \cdots\ i)$, where $2 \leqslant i < \Delta$. In a configuration of this group, the fibre at the left end and the fibre at the right end are numbered identically, meaning the two fibres break at the same moment. For example, the configuration (2 1 2) is in group α for $\Delta = 3$; (3 1 2 3) is in group α for $\Delta = 4$.

Group β is the set of configurations that have the form $(i\ i-1\ \cdots\ j)$ or the form $(j\ \cdots\ i-1\ i)$, where $2 < i \leqslant \Delta$ and $1 \leqslant j < i-1$. In a configuration of this group, the fibre numbered with i and the fibre numbered with $i-1$ must be neighbours and they also must be at one of the two ends of the configuration; the other end of the configuration is a fibre numbered with j and $1 \leqslant j < i-1$. Examples are (1 2 3) for $\Delta = 3$, (2 1 3 4) for $\Delta = 4$ and (4 3 2 1 2) for $\Delta = 5$.

Group γ is the set of configurations that have the form $(i\ j\ \cdots\ i-1)$ or the form $(i-1\ \cdots\ j\ i)$, where $2 < i \leqslant \Delta$ and $1 \leqslant j \leqslant i-1$. In a configuration of this group, the fibre numbered with i and a fibre numbered with $i-1$ are at the two ends of the configurations, one at each end. For example, (2 1 3) for $\Delta = 3$, (3 1 2 4) for $\Delta = 4$. There might be another fibre which is also numbered with $i-1$, but this fibre must be neighbour of the fibre i. Examples are (2 1 2 3) for $\Delta = 4$ and (4 3 1 2 3) for $\Delta = 5$. These kinds of configurations are also classified into group γ.

With the above classification, we can then calculate the expected number of burst events $D(\Delta)/N$ by adding up the probabilities of the three groups of configurations. In calculating the group probabilities for Δ, we find they can be calculated by using the results for $\Delta - 1$, because the probability expression for a Δ configuration contains some common factors with that of a $\Delta - 1$ configuration. Here we present an example. The probability of the configuration (2 1 2 3) for $\Delta = 4$ is

$$P(2123) = \int_0^{x_c} \mathrm{d}x_1\ p(x_1) \left[\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2'\ p(x_2') \int_{x_1}^{\frac{5}{2}x_1} \mathrm{d}x_3\ p(x_3) \right]. \tag{21}$$

Comparing equation (21) with (19), we see that the bracket in the expression for $P(2123)$ contains two integrations that are also contained in $P(212)$.

We made some calculations and found that the expected number of bursts with size Δ has the following form:

$$\frac{D(\Delta)}{N} = \frac{x_c^{\Delta\nu}}{\Delta}(C_\Delta^\alpha + C_\Delta^\beta + C_\Delta^\gamma) \tag{22}$$

where C_Δ^α, C_Δ^β and C_Δ^γ are coefficients that satisfy the following hierarchy relations:

$$\begin{cases} C_{\Delta+1}^\alpha = \dfrac{(\Delta+1)^\nu - \Delta^\nu}{2^{\nu+1}} C_\Delta^\beta + \dfrac{(\Delta+1)^\nu - 2^\nu}{2^{\nu+1}} C_\Delta^\gamma \\ C_{\Delta+1}^\beta = \dfrac{(\Delta+2)^\nu - 2^\nu}{2^\nu}(C_\Delta^\beta + C_\Delta^\gamma) \\ C_{\Delta+1}^\gamma = \dfrac{(\Delta+2)^\nu - 2^\nu}{2^{\nu-1}} C_\Delta^\alpha + \dfrac{(\Delta+2)^\nu - (\Delta+1)^\nu}{2^\nu}(C_\Delta^\beta + C_\Delta^\gamma) \end{cases} \tag{23}$$

with

$$\begin{array}{lll} C_1^\alpha = 1 & C_1^\beta = 0 & C_1^\gamma = 0 \\ C_2^\alpha = 0 & C_2^\beta = 0 & C_2^\gamma = 2\left[\left(\frac{3}{2}\right)^\nu - 1\right]. \end{array} \tag{24}$$

The coefficients C_Δ^α, C_Δ^β and C_Δ^γ we introduce into the calculations are of *critical* importance. We include in the appendix some more detailed discussions to show how the hierarchy relations are deduced for $\Delta = 1, 2, 3$ and 4.

In equation (22), C_Δ^α, C_Δ^β and C_Δ^γ can be obtained from the hierarchy relations (23) and (24). However, x_c is not obtained analytically, but from numerical calculations. The relation between x_c and N is expressed as (11) and (15). Comparing the results (23), (24) with the numerical data in figure 4, we see that the agreement is good. In deducing the burst-size distribution $D(\Delta)$, we have neglected the influence of the failed fibre on the burst

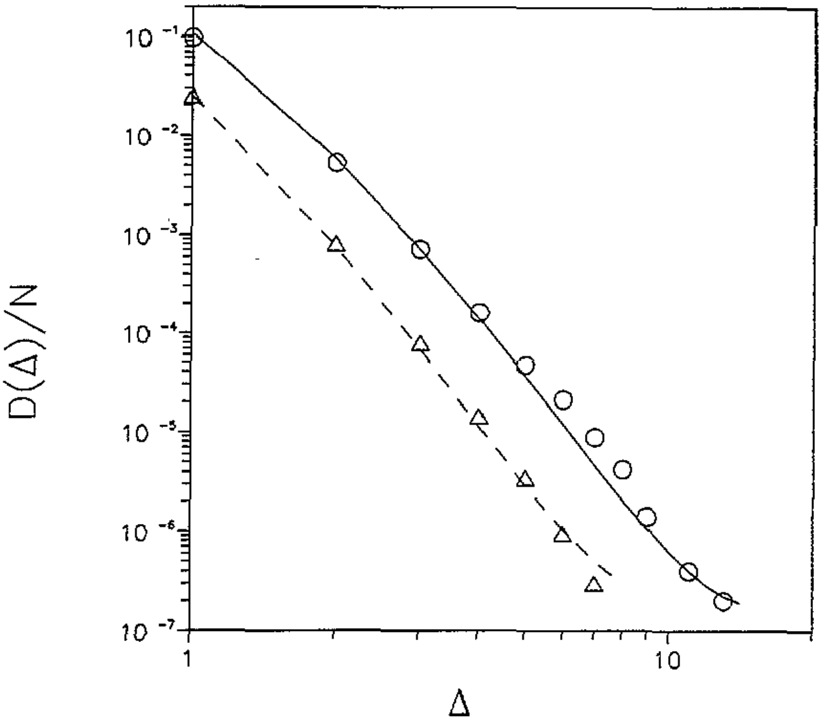

Figure 4. The burst-size distributions for a strength distribution of the form $p(x) = \nu x^{\nu-1}$ The numerical data are shown as open circles or open squares, while the analytical results (22) are shown as curves. Open circles and solid curve, $\nu = 1$, $N = 30\,000$; the power-law exponent for this case is $\xi = 5.05$. Open triangles and dashed curve, $\nu = 2$, $N = 100\,000$; the exponent for this case is $\xi = 5.80$.

events that occur later. This fact may explain the small deviation of the numerical data from the analytical results in figure 4. We did not take this into account in our calculations, partly for simplicity, but partly because this influence should not be very strong. The results of our simulations justified our approach.

We can also see that there must be some restrictions on the burst size Δ. Since we must have $(\Delta + 1)x_1/2 \leqslant 1$, and x_1 is of the order of x_c, it follows that $\Delta \leqslant 2/x_c$. This means that the formula (22) is asymptotic: with smaller x_c(larger N), it holds for larger Δ.

The asymptotic form of $D(\Delta)/N$ is not a power-law distribution. However, to a certain degree accuracy it can be approximated by $D(\Delta)/N \propto \Delta^{-\xi}$. As far as the exponent ξ is concerned, we do not find a universal value. Instead, we find that ξ increases with increasing N. The results are shown in figure 5. Note that when N is about 10^3, we have $\xi \simeq 4.5$, as in [12]. We conclude that for the local load-sharing fibre-bundle model, the exponent ξ in the power law is not universal. It depends on the size of the system as well as on the strength distribution. In the limit $N \to \infty$, the asymptotic behaviour of the system is determined by the first term in the expansion of $p(x)$ at $x = 0$. Hence, for a strength distribution of the form (16), each value of ν defines a class of 'universality', or a class of asymptotic behaviour. For example, $p(x) = 1$ and $p(x) = 2(1 - x)$ belong to the class of $\nu = 1$. From figure 5 we can see that systems of the same class have a similar tendency as $N \to \infty$. For sufficiently large N the exponent ξ increases as ν increases.

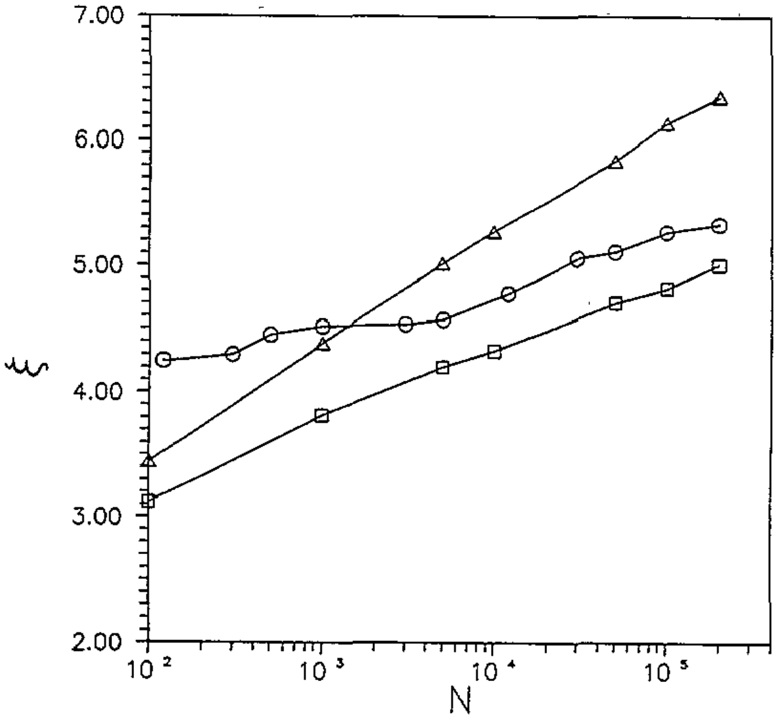

Figure 5. The exponent ξ is not universal. It depends not only on the form of the fibre strength distribution, but also on the size N of the system. The results for three forms of strength distribution are shown: ○, $p(x) = 1$; □, $p(x) = 2(1 - x)$; △, $p(x) = 2x$.

5. Discussion

In this paper we have discussed two extreme cases of fibre-bundle models, the global load-sharing and the extreme local load-sharing model. For the global load-sharing case, we are mainly concerned with the new simulation method, which greatly reduces CPU time expenditure and memory required. For the local load-sharing model, we showed that the critical load per fibre $x_c \to 0$ as the system size $N \to \infty$. Our analytical treatment reveals that the burst size distribution for the local load-sharing model is not exactly a power law, but can be approximated by a power-law relation. The exponent ξ is not universal. It depends on the strength distribution as well as on the size of the system.

The intermediate case between the global model and the local model is that where the load of the failed fibres is taken up partly by their neighbours and partly by all other elements in the system. We can introduce a quantity $\lambda \in [0, 1]$, and define the model in the following way: once a fibre fails, the load carried by this fibre will be shared by the surviving fibres. A fraction λ of the load of the broken fibre will be equally shared by its two nearest neighbours, and the remaining fraction $1 - \lambda$ of the load will be shared equally by all the unbroken fibres (including its two nearest neighbours). So, when $\lambda = 0$, the load of the broken fibre is distributed equally among all surviving fibres, thus making this case identical to that of the global load-sharing model. If $\lambda = 1$, the load of the failed fibres is taken up only by its two nearest neighbours, and the model reduces to the case of extreme local load-sharing as in section 4. It would be interesting to investigate the changes of the exponent ξ with changing λ, from 0 to 1. For this crossover model, the simulation and analytical treatment will be more complex than both extreme cases. However, we still expect the new simulation method to be applicable to this case.

Appendix

In this appendix we discuss in a little more detail the deviation of the hierarchy relations (23). As we mentioned in the text, the probability of a Δ configuration can be calculated on the basis of an appropriate $\Delta - 1$ configuration.

$\Delta = 1$

Let us first consider the probability of burst with size $\Delta = 1$. There is only one burst configuration, (1), for $\Delta = 1$, which is classified into group α. Group β and group γ are empty sets for $\Delta = 1$. Assume that the burst event (1) takes place at the fibre with strength x_1. The strengths of its two nearest neighbours must be larger than $3x_1/2$, so the probability is

$$P(1) = \int_0^{x_c} p(x_1)\mathrm{d}x_1 \left[\int_{\frac{3}{2}x_1}^{1} p(x)\,\mathrm{d}x\right]^2. \tag{A1}$$

When $N \to \infty$, $x_c \to 0$, so the factor $[\int_{3x_1/2}^{1} p(x)\,\mathrm{d}x]^2 \simeq 1$. We may omit this factor and just rewrite (A1) as

$$P(1) = \int_0^{x_c} p(x_1)\,\mathrm{d}x_1 = x_c^{\nu}. \tag{A2}$$

$\Delta = 2$

For the burst events of size $\Delta = 2$, there are two configurations (1 2) and (2 1), which are all classified into group γ. For $\Delta = 2$, groups α and β are empty sets. In the burst event (1 2), fibre 1 fails at first, so the strength of fibre 1 must be in the region $[0, x_c]$. After fibre 1 fails, fibre 2, the right neighbour of fibre 1 breaks, thus, the strength of fibre 2 must be in $[x_1, \frac{3}{2}x_1]$. Since this is a burst of size $\Delta = 2$, the left neighbour of fibre 1 and the right neighbour of fibre 2 did not break, so their strength must be in $[2x_1, 1]$. So the probability $P(12)$ of the configuration (1 2) is

$$P(12) = \int_0^{x_c} p(x_1)\,\mathrm{d}x_1 \int_{x_1}^{\frac{3}{2}x_1} p(x_2)\,\mathrm{d}x_2 \left[\int_{2x_1}^{1} p(x)\,\mathrm{d}x\right]^2. \tag{A3}$$

Here we also have $[\int_{2x_1}^{1} p(x)\,\mathrm{d}x]^2 \simeq 1$ when $N \to \infty$. So we may rewrite the above equation as

$$P(12) = \int_0^{x_c} p(x_1)\,\mathrm{d}x_1 \left[\int_{x_1}^{\frac{3}{2}x_1} p(x_2)\,\mathrm{d}x_2\right]. \tag{A4}$$

We can also write

$$P(21) = \int_0^{x_c} p(x_1)\,\mathrm{d}x_1 \left[\int_{x_1}^{\frac{3}{2}x_1} p(x_2)\,\mathrm{d}x_2\right] = P(12). \tag{A5}$$

So, $D(2)$, the expected number of bursts with size $\Delta = 2$, is

$$\frac{D(2)}{N} = P(12) + P(21) = 2P(12)$$
$$= 2\int_0^{x_c} p(x_1)\,\mathrm{d}x_1 \int_{x_1}^{\frac{3}{2}x_1} p(x_2)\,\mathrm{d}x_2$$
$$= C_2^{\gamma}\int_0^{x_c} \nu x_1^{2\nu-1}\,\mathrm{d}x_1 \qquad \text{(A6)}$$

where

$$C_2^{\gamma} = 2\left[\left(\tfrac{3}{2}\right)^{\nu} - 1\right]. \qquad \text{(A7)}$$

Here we have introduced an coefficient C_2^{γ} that will be used in the hierarchy relations to calculate the coefficients for $\Delta = 3$.

Expressing $D(2)/N$ in term of C_2^{γ}, we have

$$\frac{D(2)}{N} = \frac{x_c^{2\nu}}{2}C_2^{\gamma}. \qquad \text{(A8)}$$

$\Delta = 3$

For the burst events of size $\Delta = 3$, the five configurations are: (2 1 2), (1 2 3), (3 2 1), (2 1 3) and (3 1 2). We also classify these configurations into three groups: group α includes (2 1 2); group β includes (1 2 3) and (3 2 1); group γ include (3 1 2) and (2 1 3). The sum of the probabilities of configurations in group α is

$$P_{\alpha} = P(212) = \int_0^{x_c}\mathrm{d}x_1\, p(x_1)\left[\int_{x_1}^{\frac{3}{2}x_1}\mathrm{d}x_2\, p(x_2)\int_{x_1}^{\frac{3}{2}x_1}\mathrm{d}x_2'\, p(x_2')\right] \equiv C_3^{\alpha}\int_0^{x_c}\nu x_1^{3\nu-1}\,\mathrm{d}x_1. \qquad \text{(A9)}$$

Comparing equation (A9) with (A4), we note that in the brackets in the expressions $P(212)$ and $P(12)$ contain a common factor. This fact connects the coefficents C_2^{γ} and C_3^{α} with the following relation:

$$C_3^{\alpha} = \tfrac{1}{2}\left[\left(\tfrac{3}{2}\right)^{\nu} - 1\right]C_2^{\gamma}. \qquad \text{(A10)}$$

The probabilities for burst events in group β are

$$P(123) = P(321) = \int_0^{x_c}\mathrm{d}x_1\, p(x_1)\left[\int_{x_1}^{\frac{3}{2}x_1}\mathrm{d}x_2\, p(x_2)\int_{x_1}^{2x_1}\mathrm{d}x_3\, p(x_3)\right]. \qquad \text{(A11)}$$

The sum probability of this group is

$$P_{\beta} = P(123) + P(321) = 2P(123) = C_3^{\beta}\int_0^{x_c}\nu x_1^{3\nu-1}\,\mathrm{d}x_1. \qquad \text{(A12)}$$

We also note that the brackets in the expressions for $P(123)$ and $P(321)$ contain some factors appearing in the expressions for $P(12)$ and $P(21)$ (see equations (A4) and (A5)). Then, in a manner similar to that used to obtain (A10), we find the relation between C_3^{β} and C_2^{γ}, that is

$$C_3^{\beta} = (2^{\nu} - 1)C_2^{\gamma}. \qquad \text{(A13)}$$

Finally, the sum probability of group γ is

$$\begin{aligned} P_\gamma &= P(213) + P(312) \\ &= 2P(213) \\ &= 2\int_0^{x_c} \mathrm{d}x_1\ p(x_1) \left[\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \int_{\frac{3}{2}x_1}^{2x_1} \mathrm{d}x_3\ p(x_3) \right] \\ &= C_3^\gamma \int_0^{x_c} \nu x_1^{3\nu-1}\,\mathrm{d}x_1 \end{aligned} \tag{A14}$$

where

$$C_3^\gamma = \left[2^\nu - \left(\tfrac{3}{2}\right)^\nu\right] C_2^\gamma. \tag{A15}$$

So the expected number of bursts with size $\Delta = 3$, $D(3)$, expressed in term of the three coefficients, is

$$\frac{D(3)}{N} = P_\alpha + P_\beta + P_\gamma = \frac{x_c^{3\nu}}{3}(C_3^\alpha + C_3^\beta + C_3^\gamma). \tag{A16}$$

$\Delta = 4$

For the burst events of size $\Delta = 4$, there are 12 possible configurations. We also classify them into three groups. Group α include: (3 1 2 3) and (3 2 1 3). Group β include: (1 2 3 4), (4 3 2 1), (4 3 1 2) and (2 1 3 4). Group γ include: (4 1 2 3), (3 1 2 4), (4 2 1 3), (3 2 1 4), (2 1 2 3) and (3 2 1 2).

The probability of the burst (3 1 2 3) is

$$\begin{aligned} P(3123) &= \int_0^{x_c} \mathrm{d}x_1\ p(x_1) \int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \left[\int_{\frac{3}{2}x_1}^{2x_1} \mathrm{d}x_3\ p(x_3) \int_{x_1}^{2x_1} \mathrm{d}x_3'\ p(x_3') \right] \\ &= \frac{1}{2}\int_0^{x_c} \nu x_1^{3\nu-1}\mathrm{d}x_1\ C_3^\gamma (2^\nu - 1) x_1^\nu. \end{aligned} \tag{A17}$$

The probability of the burst (3 2 1 3) is

$$\begin{aligned} P(3213) &= \int_0^{x_c} \mathrm{d}x_1\ p(x_1) \left[\int_{x_1}^{\frac{3}{2}x_1} \mathrm{d}x_2\ p(x_2) \int_{x_1}^{2x_1} \mathrm{d}x_3\ p(x_3) \int_{\frac{3}{2}x_1}^{2x_1} \mathrm{d}x_3'\ p(x_3') \right] \\ &= \frac{1}{2}\int_0^{x_c} \nu x_1^{3\nu-1}\mathrm{d}x_1\ C_3^\beta \left[2^\nu - \left(\tfrac{3}{2}\right)^\nu\right] x_1^\nu. \end{aligned} \tag{A18}$$

The sum probability of group α is then

$$P(3213) + P(3123) = \int_0^{x_c} \nu x_1^{4\nu-1}\mathrm{d}x_1\ C_4^\alpha. \tag{A19}$$

Here we also introduce an coefficient C_4^α where

$$C_4^\alpha = \tfrac{1}{2}\left[2^\nu - \left(\tfrac{3}{2}\right)^\nu\right] C_3^\beta + \tfrac{1}{2}(2^\nu - 1) C_3^\gamma. \tag{A20}$$

In a similar way we get the sum probability for group β:

$$P(1234) + P(4321) + P(4312) + P(2134) = \int_0^{x_c} \nu x_1^{4\nu-1}\mathrm{d}x_1\ C_4^\beta \tag{A21}$$

where

$$C_4^\beta = \left[\left(\tfrac{5}{2}\right)^\nu - 1\right](C_3^\beta + C_3^\gamma). \tag{A22}$$

Likewise for group γ

$$P(4123) + P(3124) + P(4213) + P(3214) + P(2123) + P(3212)$$
$$= \int_0^{x_c} \nu x_1^{4\nu-1} \mathrm{d}x_1 \, C_4^\gamma \tag{A23}$$

where

$$C_4^\gamma = 2\left[\left(\tfrac{5}{2}\right)^\nu - 1\right] C_3^\alpha + \left[\left(\tfrac{5}{2}\right)^\nu - 2^\nu\right](C_3^\beta + C_3^\gamma). \tag{A24}$$

Then the expected number of bursts with size $\Delta = 4$, $D(4)$ can also be expressed in term of the three coefficients C_4^α, C_4^β and C_4^γ as

$$\frac{D(4)}{N} = \frac{x_c^{4\nu}}{4}(C_4^\alpha + C_4^\beta + C_4^\gamma). \tag{A25}$$

By now we have finished the calculations for $\Delta = 1$, 2, 3 and 4. We see that $D(1)/N$, $D(2)/N$, $D(3)/N$ and $D(4)/N$ all can be expressed in the form (22). More importantly, the coefficients for a given Δ can be calculated from the coefficients for $\Delta - 1$. It is then not difficult to get the hierarchy relations (23) via some general calculations.

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